Name: ____________________

Year 11 2023 Mathematics Extension 1 Assessment Task 1

Investigative Assignment with Validation Task

Task number: 1	**Weighting:** 30%	**Due Date:** Wednesday 22/3/2023

Outcomes assessed:

ME 11-5 Uses concepts of permutations and combinations to solve problems involving counting
ME 11-7 Communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Nature and description of the task:

As a result of completing this Investigative Assignment, students should be familiar with counting techniques and factorial notation. They should be able to recognise the difference between a permutation and a combination and solve relevant practical questions. Students should also be able to demonstrate the use of the binomial theorem and solve inequalities.

On the 22nd March, 2023 you will receive a selection of questions similar to the Preparation Activity below to complete in a 1 hour in-class Validation Task. You are expected to investigate/attempt each of these questions before the in-class Validation Task. The final mark for this assessment will be the mark you receive in the in-class Validation task.
NOTE: You will not have to hand in the answers to the questions in this Preparation Activity AND you will not have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to complete it on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity – investigate/attempt each of the following questions in preparation for the in-class Validation Task.

Permutations and Combinations:

1 How many ways can 11 people line up in a queue?

2 Simplify:

a $\frac{11!}{9!}$ b $\frac{(n-3)!}{(n-5)!}$ c $(k-2)!-(k+1)!$

3 Use your calculator to evaluate:

a ${}^{8}C_3$ b ${}^{11}C_4 \div {}^{8}C_4$ c $\binom{21}{19}$

4 How many 5 letter words with no repeated letters can be formed from the letters of COMBINES?

5 A number plate in a certain country consists of 2 letters A-Z followed by 4 digits 0-9. How many number plates are possible?

6 Find many arrangements of the letters of the word REMINISE are possible:

a if the vowels and consonants must alternate,

b if the word must start with M and end with N,

c if all the consonants must be in a group at the end of the word,

d if the R is somewhere to the right of the M.

7 6 boys and 7 girls are to sit in a row. Find how many ways this can be done if:

a there are no restrictions,

b the boys and girls sit in distinct groups,

c a particular boy and girl must sit together.

8 How many ways can the letters of COMMITTEE be arranged?

9 How many words of three or four letters may be formed using the letters of AGUSTIN?

10 A quiz consists of 20 questions, each taking the answer Yes or No. How many ways is it possible to get 13 correct and 7 incorrect answers?

11 A committee of 6 is chosen from 8 men and 7 women. Find how many committees are possible if:

a there are no restrictions

b all members are to be female,

c all members are to be male,

d there are to be exactly two men,

e there are to be equal amounts of men and women,

f there is to be a majority of women,

g a particular man must be included,

h a particular man must not be included,

i Michel refuses to be on the committee with Clark.

12 Twelve people arrive at a restaurant. How many ways can they be assigned to:

a a table for five and a table for seven,

b two indistinguishable tables for six?

13 What is the probability that if a committee of five is formed at random from five men and three women, it will have more men and women?

14 From a standard pack of 52 cards, three are selected at random. Find the probability that:

a they are the jack of diamonds, four of spades and king of hearts,

b all three are fives,

c they are all diamonds or hearts,

d none of them are spades,

e one is red and two are black,

f they are all picture cards,

g two are sevens and one is a picture card,

h at most two of them are hearts.

15 Four boys and four girls are arranged in a circle. In how many ways can this be done:

a If there are no restrictions,

b if the boys and the girls alternate,

c if the boys and girls are in distinct groups,

d if a particular boy and girl wish to sit next to one another,

e if two particular boys do not wish to sit next to one another,

f if a particular boy wants to sit opposite a particular girl?

16 Twenty-two peanuts are shared amongst seven monkeys. Show that there is at least one monkey who receives at least four peanuts.

17 Amongst 2000 people, show that there is guaranteed to be a birthday shared by at least 6 people.

Answers

1 39916800

2 a 110

b $(n-3)(n-4)$

c $(1-k^3+k)(k-2)!$

3 a 56

b $\frac{33}{7}$

c 210

4 6720

5 6760000

6 a 288

b 180

c 144

d 5040

7 a 6227020800

b 7257600

c 958003200

8 45360

9 1050

10 77520

11 a 5005

b 7

c 28

d 980

e 1960

f 1155

g 2002

h 3003

i 2574

12 a 792

b 462

13 $\frac{23}{28}$

14 a $\frac{1}{22100}$

b $\frac{1}{5525}$

c $\frac{2}{17}$

d $\frac{703}{1700}$

e $\frac{13}{34}$

f $\frac{11}{1105}$

g $\frac{1}{5525}$

h $\frac{839}{850}$

15 a 5040

b 144

c 144

d 1440

e 3600

f 720

16 By the pigeon-hole principal. 21 peanuts can be split evenly between 7 monkeys, with all monkeys having exactly 3 peanuts. The final peanut must go to one of the monkeys, hence why there is at least one monkey who receives at least four peanuts.

17 By the pigeon-hole principal.

Binomial Theorem:

1 Write out the first eight rows of Pascal's triangle,

2 Use your answer to the previous question to expand:

a $(1-x)^6$ **b** $(2+x)^5$ **c** $(x-y)^5$ **d** $(2x-yz)^4$

3 **a** Expand $(1-5x)^5$ as far as the term in x^2.

b Hence find the coefficient of x^2 in the expansion of $(3+2x)(1-5x)^5$.

4 **a** Expand $(1+x)^8$

b Hence find the first decimal place of 1.04^8.

5 Expand:

a $(5-3x)^4$ **b** $(4y-3)^5$ **c** $(3x+2y)^3$ **d** $\left(2x-\frac{1}{x^2}\right)^4$

6 Use the result $nC_r = \frac{n!}{r!(n-r)!}$ To evaluate the following. Do not use a calculator.

a $6C_4$ **b** $8C_3$ **c** $9C_6$ **d** $100C_{30} \div 101C_{30}$

7 Use your calculator button $[nC_r]$ to evaluate the following.

a $12C_7$ **b** $\binom{14}{6}$ **c** $7C_3$ **d** $\binom{8}{5} \div \binom{5}{3}$

e $\frac{12C_7}{7C_4}$ **f** $\frac{15C_7}{10C_3}$

8 Use your understanding of the patterns in Pascal's triangle to simplify the following expressions:

a $0C_0$ **b** $44C_4 - 44C_{40}$

c $4C_0 + 4C_1 + 4C_2 + 4C_3 + 4C_4$ **d** $2n+1C_n - 2nC_n - 2nC_{n-1}$

9 Solve $nC_2 = 21$.

10 Find the coefficient of x^7 in the expansion of $(2+x)(1-4x)^{12}$.

11 **a** Write out the first few terms in the expansion of $(1-x)^n$.

b By an appropriate substitution, prove that:

$$\binom{n}{0} - 3 \times \binom{n}{1} + 9 \times \binom{n}{2} - 27 \times \binom{n}{3} + \cdots = (-2)^n$$

c Verify this result for the row indexed by $n = 4$ in Pascal's triangle.

12 **a** Prove that $nC_0 + nC_1 + nC_2 + nC_3 + \cdots = 2^n$.

b What is the significance of this result for Pascal's triangle?

c How may this result be interpreted as a sum of combinations?

Answers

1

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

2 a $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

b $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$

c $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

d $16x^4 - 32x^3yz + 24x^2y^2z^2 - 8xy^3z^3 + y^4z^4$

3 a $1 - 25x + 250x^2 - \cdots$

b 700

4 a $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

b 1.3, first decimal place is 3

5 a $625 - 1500x + 1350x^2 - 540x^3 + 81x^4$

b $1024y^5 - 3840y^4 + 5760y^3 - 4320y^2 + 1620y - 243$

c $27x^3 + 54x^2y + 36xy^2 + 8y^3$

d $16x^4 - 32x + \frac{24}{x^2} - \frac{8}{x^5} + \frac{1}{x^8}$

6 a 15

b 56

c 84

d $\frac{71}{101}$

7 a 792

b 3003

c 35

d $\frac{28}{5}$

e $\frac{792}{35}$

f $\frac{429}{8}$

8 a 1

b 0

c 16

d 0

9 $n = 7$

10 -22167552

11 a $\binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots$

b Recognise that the substitution $x = 3$ is needed.

$\binom{n}{0} - \binom{n}{1} \times 3 + \binom{n}{2} \times (3)^2 - \binom{n}{3} \times (3)^3 + \cdots = (1-3)^n$.

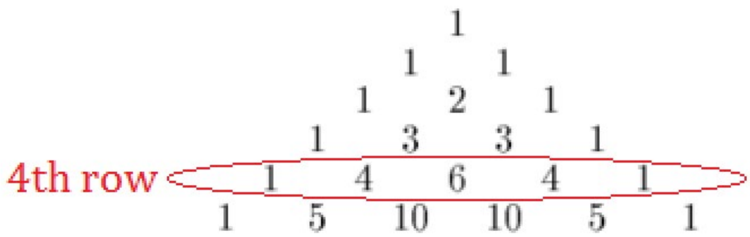
$\binom{n}{0} - 3 \times \binom{n}{1} + 9 \times \binom{n}{2} - 27 \times \binom{n}{3} + \cdots = (-2)^n$.

c

We can see that the result corresponds to the 4th row of the Pascal triangle when a substitution of $n = 4$ is made for the equation found in part a.

$\binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots$

$1 - 4x + 6x^2 - 4x^3 + 1$.



12 a Expand $(1 + x)^n$, then make the substitution $x = 1$.

b This shows that the sum of values in row n of the Pascal's triangle for $n = 0, 1, 2, 3, \dots$. will equate to

2 to the power of that row.

c Each individual value of in any row of a Pascal's triangle can be represented by nC_r, where n represents the row number ($n = 0, 1, 2, 3, \dots$) and r represents which value of the row counting from the left ($r = 0,1,2,3, \dots$). Hence the sum of the rows of a Pascal's triangle can be interpreted as a sum of combinations.

Inequalities:

1 Solve each inequality.

a $x^2 - 12x + 32 \geq 0$ b $4x > x^2$ c $8x \leq 3x^2 - 35$

2 Solve each inequality,

a $|x| \geq 5$ b $|x - 3| < 6$ c $|4x - 3| > 11$

3 Solve the inequality.

a $\frac{2}{x} < 5$ **b** $\frac{4}{x+5} \geq 3$ **c** $\frac{2x-1}{3x+2} < 1$

Answers

1

a $x \leq 4, x \geq 8$ **b** $0 < x < 4$ **c** $x \leq -\frac{7}{3}, \quad x \geq 5$

2

a $x \leq -5, x \geq 5$ **b** $-3 < x < 9$ **c** $x < -2, x > \frac{7}{2}$

3

a $x < 0, x > \frac{2}{5}$ **b** $-5 < x \leq -\frac{11}{3}$ **c** $x < -3, x > -\frac{2}{3}$

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

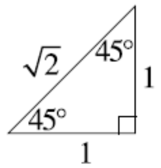
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

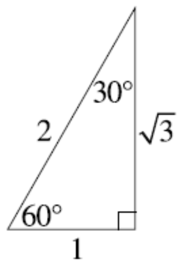


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\text{cosec}\, A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

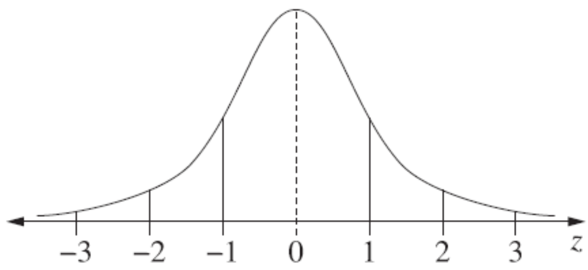
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$E(X) = \mu$

$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \leq x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \text{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\text{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$