Name: ____________________

Year 12 2023 Mathematics Advanced Assessment Task 3

Assignment with Validation Task		
Task number: 3	**Weighting:** 25%	**Due Date:** Thursday 25/5/23

Outcomes assessed:

MA 12-1	uses detailed algebraic and graphical techniques to critically construct, model and evaluate arguments in a range of familiar and unfamiliar contexts
MA 12-3	applies calculus techniques to model and solve problems
MA 12-6	applies appropriate differentiation methods to solve problems
MA 12-7	applies the concepts and techniques of indefinite and definite integrals in the solution of problems
MA12-10	constructs arguments to prove and justify results and provides reasoning to support conclusions which are appropriate to the context

Nature and description of the task:

As a result of completing this Assignment, students should be familiar with all content related to the following topics:

- Further Differentiation (Chapter 4 of the Advanced Grove Book and Chapter 5 of the Extension 1 Grove Book).
- Geometrical Applications of Differentiation (Chapter 5 of the Advanced Grove Book and Chapter 6 of the Extension 1 Grove Book).
- Integration (Chapter 6 of the Advanced Grove Book and Chapter 7 of the Extension 1 Grove Book).

On the 25th May, 2023 you will receive a selection of similar questions to the Preparation Activity below to complete in an in-class Validation Task. You are expected to investigate/attempt each of these questions before the in-class Validation Task. The final mark for this assessment will be the mark you receive in the in-class Validation task. NOTE: You will not have to hand in the answers to the questions in this Preparation Activity AND you will not have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to complete it on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity – investigate/attempt each of the following questions in preparation for the in-class Validation Task.

Further Differentiation:

1 Find the derivative of each function.

a $f(x) = x^2 - 2x$

b $y = \dfrac{1}{x}$

c $y = \dfrac{x}{x^2 - 1}$

d $g(x) = (x^2 - x)^2$

e $h(x) = 4^x$

f $f(x) = 2xe^{x^2}$

g $f(x) = \log_e x^3$

2 Find the anti-derivative of each function.

a $f(x) = 4x^3 - 3x^2 + x$

b $\dfrac{dy}{dx} = (5x - 7)^3$

c $g(x) = \tan^2 x \sec^2 x$

d $h(x) = \dfrac{4x - 1}{2x^2 - x}$

e $\dfrac{du}{dx} = xe^{x^2}$

f $y' = 2x^3 - 2x \sin x^2 \cos x^2$

3 Find the equation of the normal to the graph for the function $y = x^3 + 2 \ln x^2$ when $x = -1$.

4 A hyperbola has equation $y=\frac{2}{x}$. Find the equation of the tangent to the hyperbola at $x = 2$, in general form.

5 Find the exact gradient of the tangent to the function $f(x) = e^{\ln x^2}$ at $x = \frac{1}{2}$.

6 Find the equation of the function $f(x)$ if $f'(x)=xe^{x^2}$ and the point $P\left(0,\frac{5}{2}\right)$ lies on the graph of the function.

7 Find $y = f(x)$ if $f''(x) = 3x^2 - 6x + 2$, $f'(2) = -3$ and $f(-1) = 2$.

8 Find the derivative of $y = \sin 2x°$.

9 The rate of change of volume V with respect to time (t) is $\frac{dV}{dt}=(2t-1)^3$.

If $V = 3$ when $t = 0$, find the volume when $t = 3$.

10 Find the second derivative of $v=\frac{x}{x^2-5}$.

11 Find each integral.

a $\int \sqrt{x}+\frac{2}{x^2}+e^{3-x}dx$

b $\int 3x\sec^2(3x^2)dx$

c $\int -\sin x\,dx$

12 A particle has velocity $v=\frac{t}{t^2+1}$ m s^{-1}. Find its displacement after 4 s if its initial displacement is 3 m.

13 A curve has its rate of change given by $\frac{dy}{dx}=\sin(3x)$ and passes through the point $\left(\frac{\pi}{2},\pi\right)$.

Find the equation of the curve.

Geometrical Applications of Differentiation:

1 For what values of x is the graph of $y = 3x^2 - 4x + 1$ increasing?

2 The curve $f(x) = x^3 + ax^2 + bx - 4$ has stationary points at $x = 3$ and $x = 5$. Find the values of a and b.

3 Draw a sketch of a function $y = f(x)$ where $\frac{dy}{dx} > 0$ for $x < 5$, $\frac{dy}{dx} = 0$ when $x = 5$ and $\frac{dy}{dx} < 0$ for $x > 5$.

4 Consider the function $f(x) = 2x^3 + 6x^2 - 1$ in the domain $[-3, 1]$.

a State the y-intercept.

b Locate any stationary points and determine their nature.

c Determine the point of inflection.

d Hence sketch the graph of $f(x) = 2x^3 + 6x^2 - 1$ in the domain $[-3, 1]$.

e What is the global maximum value of the function in this domain?

5 Consider the graph of the function $y = 2x^3 - 6x^2 + 6x + 1$.

a Show that the graph has only one stationary point, find its coordinates and determine its nature.

b State the values of x for which the curve is concave up.

c State the values of x for which the curve is increasing.

6 Consider the function $y = x\sqrt{4 - x^2}$.

a What is the domain of this function?

b Find the stationary points for this function and determine their nature.

c Sketch the graph of the function.

7 The 3 dimensions (length, breadth and height) of a box with a square base of side x all add up to 40 cm. Show that the volume of this box is given by $V = 40x^2 - 2x^3$ and hence find the maximum volume of this box.

8 A scenic helicopter flight can take x passengers per week for a charge of $(200 - 0.3x)$ dollars per person. The number of flights is not relevant. The cost of operation each week is \$$(3000 + 50x)$ dollars when x passengers are taken.

a Show that the profit, in dollars, per week is given by $P = 150x - 0.3x^2 - 3000$.

b How many passengers should be taken, per week, to maximise the profit?

c Hence, given this number of passengers, what is the charge, per passenger, when this maximum profit occurs?

Integration:

1 Use the trapezoidal rule to approximate $\int_2^5 \frac{1}{x-1}\,dx$ using 4 subintervals.

2 Evaluate:

a $\int_1^2 \frac{3x^3 - x^2 + 4x}{x}\,dx$

b $\int_2^3 (4x+2)\sqrt{x^2+x-4}\,dx$

3 Find the change in displacement between 1 and 3 seconds if the velocity function is $v = 6t^2 - 8t + 1$ cm s^{-1}.

4 Find each integral.

a $\int \sqrt{x} + \frac{2}{x^2} + e^{3-x}\,dx$

b $\int 3x\sec^2(3x^2)\,dx$

c $\int -\sin x\,dx$

5 **a** Show that $\frac{3x-4}{x^2-16} = \frac{1}{x-4} + \frac{2}{x+4}$.

b Hence find $\int \frac{3x-4}{x^2-16}\,dx$.

6 A particle has velocity $v = \frac{t}{t^2+1}$ m s^{-1}. Find its displacement after 4 s if its initial displacement is 3 m.

7 A curve has its rate of change given by $\frac{dy}{dx} = \sin(3x)$ and passes through the point $\left(\frac{\pi}{2}, \pi\right)$.

Find the equation of the curve.

8 Find the area enclosed by the curve $y = e^{3^{x}} - 1$ and the lines $y = 0$ and $x = 1$.

9 Find the area enclosed between the curves $y = x^2$, $y = (x + 2)^2$ and the x-axis.

10 **a** Sketch and shade the area bounded by the curve $y = \sqrt{x+4}$, the y-axis and the lines $y = 0$ and $y = 3$.

b Calculate the total shaded area.

End of Part 1 Preparation Activity

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

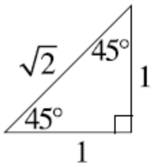
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

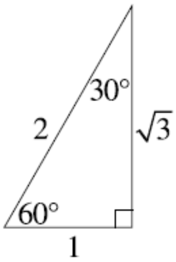


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

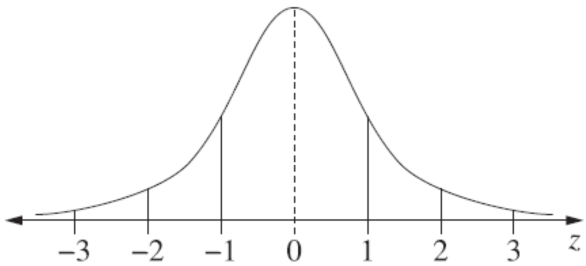
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

Further Differentiation Answers:

1 **a** $f'(x) = 2x - 2$ **b** $y' = -\frac{1}{x^2}$ **c** $y' = -\frac{x^2+1}{(x^2-1)^2}$

d $g'(x) = 2(x^2 - x)(2x - 1)$ **e** $h'(x) = 4^x \ln 4$ **f** $f'(x) = 2e^{x^2}(2x^2+1)$

g $f'(x) = \frac{3}{x}$

2 **a** $F(x) = x^4 - x^3 + \frac{x^2}{2} + c$ **b** $y = \frac{(5x-7)^4}{20} + c$ **c** $G(x) = \frac{\tan^3 x}{3} + c$

d $y = \ln|2x^2 - x| + c$ **e** $u = \frac{e^{x^2}}{2} + c$ **f** $y = \frac{x^4}{2} + \frac{\cos^2(x^2)}{2}$

3 $y = x$

4 $x + 2y - 4 = 0$

5 $m = 1$

6 $\frac{e^{x^2}}{2} + 2$

7 $f(x) = \frac{x^4}{4} - x^3 + x^2 - 3x - \frac{13}{4}$

8 $y' = \frac{\pi}{90}\cos\left(\frac{\pi x^\circ}{90}\right)$

9 81

10 $\frac{2x^3 + 30x}{(x^2-5)^3}$

11 **a** $\frac{2\sqrt{x^3}}{3} - \frac{2}{x} - e^{3-x} + C$ **b** $\frac{1}{2}\tan(3x^2) + C$ **c** $\cos x + C$

12 7.4166 m

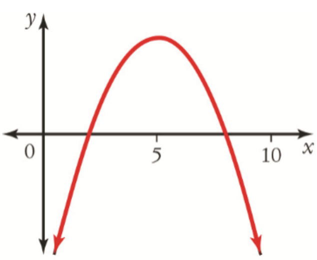
13 $y = -\frac{1}{3}\cos(3x) + \pi$

Geometrical Applications of Differentiation Answers:

1 $x > \frac{2}{3}$

2 $a = -12, b = 45$

3

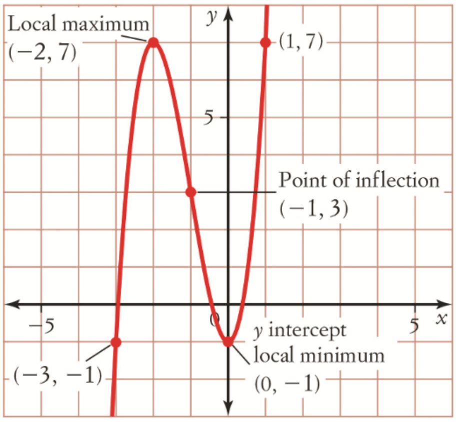


4 **a** –1

b (–2, 7) maximum, (0, –1) minimum

c (–1, 3)

d



e $y = 7$

5 **a**

$$\frac{dy}{dx} = 6x^2 - 12x + 6$$
$$= 6(x^2 - 2x + 1)$$
$$= 6(x-1)^2$$

Stationary point occurs when $\frac{dy}{dx} = 0$

$6(x-1)^2 = 0$

$\therefore x = 1$

$$\frac{d^2y}{dx^2} = 12x - 12$$
$$= 12(x-1)$$

When $x = 1$, $\frac{d^2y}{dx^2} = 0$

x	0	1	2
y''	−12	0	12

Hence one stationary point is a horizontal point of inflection (1, 7).

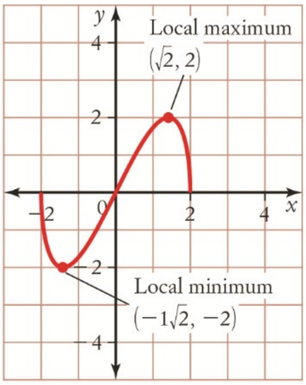
b $x > 1$

c all x except $x = 1$

6 **a** [–2, 2]

b $\left(-\sqrt{2}, -2\right)$ mimimum, $\left(\sqrt{2}, 2\right)$ maximum

c



7 $2x + h = 40$

$h = 40 - 2x$

$V = x^2h$

$= x^2(40 - 2x)$

$= 40x^2 - 2x^3$

$\frac{dV}{dx} = 80x - 8x^2$

$= 8x\,(10 - x)$

$= 0$ when $x = 0, 10$

However 0 is not a solution

$\frac{d^2V}{dx^2} = 80 - 16x$

$f''(10) = 80 - 160$

$= -80\ (<0)$

$\therefore$ Max volume occurs when $x = 10$

Max volume = $10^3 = 1000\text{ cm}^3$

8 **a** Weekly Profit = Income − Expenditure

= charge × no. of passengers

− costs of operation

$P = (200 - 0.3x)x - (3000 + 50x)$

$= 150x - 0.3x^2 - 3000$

b 250 **c** \$125

Integration Answers:

1 1.42809 units^2

2 **a** 9.5 **b** $\frac{56\sqrt{2}}{3}$

3 **a** 23 cm

4 **a** $\frac{2\sqrt{x^3}}{3} - \frac{2}{x} - e^{3-x} + C$ **b** $\frac{1}{2}\tan\left(3x^2\right) + C$ **c** $\cos x$ ▯ C

5 **a** $\text{RHS} = \dfrac{1}{x-4} + \dfrac{2}{x+4}$

$= \dfrac{1(x+4)+2(x-4)}{(x-4)(x+4)}$

$= \dfrac{x+4+2x-8}{x^2-16}$

$= \dfrac{3x-4}{x^2-16}$

$= \text{LHS}$

b $\ln|x-4| + 2\ln|x+4| + C$

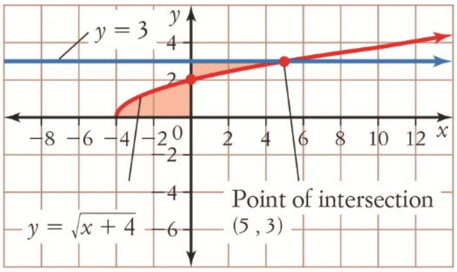
6 7.4166 m

7 $y = -\dfrac{1}{3}\cos(3x) + \pi$

8 $\dfrac{1}{3}e^3 - \dfrac{4}{3}$ units^2

9 $\dfrac{2}{3}$ units^2

10 **a**



b Area $= 7\dfrac{2}{3}$ units^2