Name: ____________________

Year 12 2023/2024 Mathematics Extension 1 Assessment Task 1

Investigative Assignment with Validation Task		
Task number: 1	**Weighting:** 20%	**Due Date:** Wednesday 6/12/23

Outcomes assessed:

ME11-3 applies concepts and techniques of inverse trigonometric functions and simplifying expressions involving compound angles in the solution of problems
ME11-4 applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change
ME12-1 applies techniques involving proof or calculus to model and solve problems
ME12-7 evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms

Nature and description of the task:

As a result of completing this Investigative Assignment, students should be familiar with inverse trigonometric functions and their graphs, simplifying trigonometric functions using sum and difference angle properties, double angle properties and t results. They should be familiar with problems involving rates of change, exponential growth and decay and related rates of change. They will also be able to apply the principles of mathematical induction to prove a variety of problems.

On the 6^{th} December, 2023 you will receive a selection of similar questions from the Preparation Activity below to complete in a 1 hour in-class Validation Task. You are expected to investigate/attempt each of these questions before the in-class Validation Task. The final mark for this assessment will be the mark you receive on the in-class Validation task. NOTE: You will not have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to hand in / complete both parts of the Assignment on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is submitted/completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity – investigate/attempt each of the following questions in preparation for the in-class Validation Task.

Further Rates:

1 The side lengths of a square are increasing at 3 mm per hour.

a Find the rate at which the area is increasing when the side length is 10 cm.

b Find the rate at which the diagonal is increasing when the area is 64 cm^2.

2 Coal is pouring from a high conveyor belt onto a conical pile. The conical pile has a semi-vertical angle of 30°, and at time t the radius of the base is r.

a Find the height of the cone in terms of r, and hence show that the volume V of the cone is $V = \dfrac{\pi r^3\sqrt{3}}{3}$. (See Box 2 for the various formulae.)

b Differentiate to express $\dfrac{dV}{dt}$ in terms of $\dfrac{dr}{dt}$.

c Find the slant height ℓ in terms of r, and hence show that the area A of the curved surface of the cone is $A = 2\pi r^2$. (See Box 2.)

d Differentiate to express $\dfrac{dA}{dt}$ in terms of $\dfrac{dr}{dt}$.

e Hence show that $\dfrac{dA}{dt} = \dfrac{4}{r\sqrt{3}}\dfrac{dV}{dt}$.

f If the coal is pouring onto the pile at a rate of 5 m^3/s, find the rate at which the radius and area are increasing when the radius is 4 metres.

3 Mice are multiplying in Mosman. Their population M was estimated to be 80000 at the start of 2010 and 130000 at the start of 2015. Assume an exponential growth model,

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant and } t \text{ is time in years after 2010.}$$

a Show that $M = 80000e^{kt}$ satisfies the differential equation, and has initial value 80 000.

b Find k, and predict the population at the start of 2028, correct to the nearest 1000.

c Predict in what year the population will first exceed 1000000.

4 Caesium-137 has a half-life of about 30.2 years, meaning that half its mass decays into something else in 30.2 years. It was deposited in very tiny, but highly toxic, glass-like grains around Fukushima in Japan as a result of the nuclear power-plant disaster there on 11th March 2010. Radioactive decay follows the exponential decay model $\dfrac{dM}{dt} = -kM$, where M is the mass remaining after time t years from an initial mass M_0.

a Prove that $M = M_0e^{-kt}$ satisfies the differential equation and has initial mass M_0.

b Use the half-life given above to calculate k.

c Find what percentage of the original caesium-137 remains after 100 years, correct to the nearest 0.01%.

d How long will it take for the mass of caesium-137 to decrease to 1% of its original level, correct to the nearest year?

5 A steel bar is taken out of a fire that has a temperature of 500°C. Newton's law of cooling tells us that the temperature T after t minutes satisfies the differential equation

$$\frac{dT}{dt} = k(T - E), \quad \text{where } k \text{ is a constant and } E \text{ is room temperature.}$$

a Explain why the constant k is negative.

b Show that $T = E + Ae^{kt}$ satisfies the differential equation, and that $A = 500 - E$.

c After 6 minutes, the bar has cooled to 250°C.

i If room temperature is 0°C, find A and k, and find the temperature after 15 minutes, correct to the nearest degree.

ii If room temperature is 40°C, find A and k, and find the temperature after 15 minutes, correct to the nearest degree.

6 Goats have been introduced for the second time onto Goat Island, and their population P is growing. It is known from earlier years that their population is limited by lack of resources to an estimated maximum of $M = 10000$. Their numbers are therefore being modelled by the differential equation

$$\frac{dP}{dt} = -k(P - M), \quad \text{where } k \text{ is a constant and } t \text{ is time in years.}$$

a Explain why the constant k is positive.

b Show that $P = M - Ae^{-kt}$ satisfies the differential equation, has initial value $M - A$, and has limit 10000.

c The population at the start of 2010 was 500, and at the start of 2020 is 2000.

i Find A and k.

ii What is the predicted population, correct to the nearest 10 goats, at the start of 2030?

iii In what year is the population predicted to reach 8000?

Box 2 – Possible Formulae:

2 VOLUME AND SURFACE AREA OF SOLIDS

A sphere:	A cylinder:	A cone:	A pyramid:
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{1}{3} \times \text{base} \times \text{height}$
$A = 4\pi r^2$	$A = 2\pi r^2 + 2\pi r h$	$A = \pi r^2 + \pi r \ell$	$A = \text{sum of faces}$

In the formula for the surface area of a cone, ℓ is the slant height.

Further Trigonometry:

1 The function $f(x) = \frac{1}{2}x + 1$ is defined for $-4 \le x \le 6$.

a Find the range of $f(x)$.

b Find the equation of $f^{-1}(x)$.

c State the domain and range of $f^{-1}(x)$.

d Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram.

e About which line are the two graphs symmetrical?

2 Consider the function $F(x) = \ln(x + 1)$.

a State the domain and range of $F(x)$.

b Find the equation of $F^{-1}(x)$.

c State the domain and range of $F^{-1}(x)$.

d Sketch $F(x)$ and $F^{-1}(x)$ on the same diagram.

e Classify $F(x)$ and $F^{-1}(x)$ as increasing, decreasing or neither.

3 Consider the function $Q(x) = (x - 2)^2$.

a Sketch the graph of $Q(x)$.

b What is the largest domain containing $x = 0$ for which $Q(x)$ has an inverse function?

c Find the equation of $Q^{-1}(x)$ corresponding to the restricted domain of $Q(x)$ in part **b**.

d Sketch $Q(x)$, with its restricted domain, and $Q^{-1}(x)$ on the same diagram.

4 Write down the exact value of:

a $\sin^{-1}1$ **b** $\cos^{-1}\frac{\sqrt{3}}{2}$ **c** $\tan^{-1}\sqrt{3}$

d $\tan^{-1}(-1)$ **e** $\cos^{-1}\left(-\frac{1}{2}\right)$ **f** $\sin^{-1}\left(-\frac{1}{2}\right)$

5 Find the exact value of:

a $\cos\left(\cos^{-1}1\right)$ **b** $\sin\left(\tan^{-1}1\right)$ **c** $\cos\left(\tan^{-1}(-\sqrt{3})\right)$

d $\tan^{-1}\left(-\tan\frac{\pi}{6}\right)$ **e** $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ **f** $\tan^{-1}\left(-2\sin\frac{2\pi}{3}\right)$

6 Use a right-angled triangle to find the exact value of:

a $\sin\left(\cos^{-1}\frac{1}{3}\right)$ **b** $\cos\left(\tan^{-1}\left(-\frac{\sqrt{7}}{3}\right)\right)$

7 Sketch the graph of each function, stating the domain and range.

a $y = -\tan^{-1}x$ **b** $y = \sin^{-1}(x + 1)$ **c** $y = \cos^{-1}(x - 1) - \pi$

8 Simplify, using the compound-angle results:

a $\cos 3\theta \cos\theta + \sin 3\theta \sin\theta$ **b** $\sin 50^\circ \cos 10^\circ - \cos 50^\circ \sin 10^\circ$

c $\dfrac{\tan 41^\circ + \tan 9^\circ}{1 - \tan 41^\circ \tan 9^\circ}$ **d** $\cos 15^\circ \cos 55^\circ - \sin 15^\circ \sin 55^\circ$

e $\sin 4\alpha \cos 2\alpha + \cos 4\alpha \sin 2\alpha$ **f** $\dfrac{1 + \tan 2\theta \tan\theta}{\tan 2\theta - \tan\theta}$

9 Simplify, using the double-angle results:

a $2\sin 2\theta \cos 2\theta$ **b** $\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$ **c** $2\cos^2 3\alpha - 1$

d $\dfrac{2\tan 35^\circ}{1 - \tan^2 35^\circ}$ **e** $1 - 2\sin^2 25^\circ$ **f** $\dfrac{2\tan 4x}{1 - \tan^2 4x}$

10 Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find:

a $\cos A$ **b** $\cos 2A$ **c** $\cos (A + B)$

d $\sin 2B$ **e** $\tan 2A$ **f** $\tan (B - A)$

11 **a** By writing $75°$ as $45° + 30°$, show that:

i $\sin 75° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ **ii** $\cos 75° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

b Hence show that:

i $\sin 75° \cos 75° = \frac{1}{4}$ **ii** $\sin 75° - \cos 75° = \sin 45°$

iii $\sin^2 75° - \cos^2 75° = \sin 60°$ **iv** $\sin^2 75° + \cos^2 75° = 1$

12 Use the compound-angle and double-angle results to find the exact value of:

a $2 \sin 15° \cos 15°$ **b** $\cos 35° \cos 5° + \sin 35° \sin 5°$

c $\dfrac{\tan 110° + \tan 25°}{1 - \tan 110° \tan 25°}$ **d** $1 - 2\sin^2 \frac{\pi}{8}$

e $\cos \frac{\pi}{12} \sin \frac{\pi}{12}$ **f** $\sin \frac{8\pi}{9} \cos \frac{2\pi}{9} - \cos \frac{8\pi}{9} \sin \frac{2\pi}{9}$

13 Prove each identity.

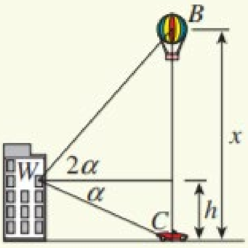
a $(\sin \alpha - \cos \alpha)^2 = 1 - \sin 2\alpha$ **b** $\cos A - \sin 2A \sin A = \cos A \cos 2A$

c $\sin 2\theta(\tan \theta + \cot \theta) = 2$ **d** $\cot \alpha \sin 2\alpha - \cos 2\alpha = 1$

e $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$ **f** $\dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta$

g $\dfrac{1}{1 - \tan A} - \dfrac{1}{1 + \tan A} = \tan 2A$ **h** $\tan 2A (\cot A - \tan A) = 2$, (provided $\cot A \neq \tan A$)

14 An office-worker is looking out a window W of a building standing on level ground. From W, a car C has an angle of depression α, while a balloon B directly above the car has an angle of elevation 2α. The height of the balloon above the car is x, and the height of the window above the ground is h.



a Show that $\dfrac{\tan \alpha}{h} = \dfrac{\tan 2\alpha}{x - h}$.

b Hence show that $\dfrac{h}{x} = \dfrac{1 - \tan^2 \alpha}{3 - \tan^2 \alpha}$.

15 **a** If $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$, show that $\tan (\alpha + \beta) = 1$.

b Hence find the exact value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

16 Show without a calculator that:

a $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$ **b** $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{2}{9}$

17 Show without a calculator that:

a $2\sin^{-1}\frac{2}{3} = \sin^{-1}\frac{4\sqrt{5}}{9}$ **b** $2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \cos^{-1}\left(-\frac{3}{5}\right)$

18 Solve each equation for $0 \leq x \leq 2\pi$.

a $\sin 2x + \cos x = 0$ **b** $\cos 2x = \sin x$

c $2\cos 2x + 8\cos x + 5 = 0$ **d** $\tan 2x = 3\tan x$

19 Write in terms of t, where $t = \tan \frac{1}{2}\theta$:

a $\sin \theta$ **b** $\cos \theta$ **c** $\tan \theta$

d $\sec \theta$ **e** $1 + \cos \theta$ **f** $\dfrac{\sin \theta}{1 + \cos \theta}$

20 Use the $t = \tan \frac{1}{2}\theta$ results to find the exact value of:

a $\dfrac{2 \tan 67\frac{1}{2}^\circ}{1 - \tan^2 67\frac{1}{2}^\circ}$ **b** $\dfrac{2 \tan 67\frac{1}{2}^\circ}{1 + \tan^2 67\frac{1}{2}^\circ}$ **c** $\dfrac{1 - \tan^2 67\frac{1}{2}^\circ}{1 + \tan^2 67\frac{1}{2}^\circ}$

21 Prove each identity using the $t = \tan \frac{1}{2}\theta$ results.

a $\sin \theta \tan \frac{1}{2}\theta + \cos \theta = 1$

b $\operatorname{cosec}\theta - \cot \theta = \tan \frac{1}{2}\theta$

c $\sin 2\alpha(\tan 2\alpha - \tan \alpha) = \tan 2\alpha \tan \alpha$

d $\dfrac{\cos 2\alpha - \sin 2\alpha + 1}{\cos 2\alpha + \sin 2\alpha - 1} = \cot \alpha$

22 **a** If $t = \tan 75^\circ$, show that $\dfrac{2t}{1 - t^2} = -\frac{1}{\sqrt{3}}$.

b **i** Hence show that $\tan 75^\circ = 2 + \sqrt{3}$.

ii What does the other root of the equation represent?

23 Use the products-to-sums identities to express as a sum or difference of two trigonometric functions:

a $2 \cos 2A \cos A$ **b** $2 \cos 5A \sin 3A$

c $2 \sin (3A + 2B) \cos (2A - B)$ **d** $2 \sin (2A + 5B) \sin (A + 3B)$

24 Use the products-to-sums identities to show that:

a $2 \cos 58^\circ \cos 32^\circ = \cos 26^\circ$ **b** $2 \cos 45^\circ \sin 35^\circ = \cos 10^\circ - \sin 10^\circ$

c $2 \sin 70^\circ \sin 50^\circ = \frac{1}{2} + \cos 20^\circ$ **d** $2 \sin 55^\circ \cos 40^\circ = \sin 15^\circ + \cos 5^\circ$

25 Prove these identities using products-to-sums formulae.

a $4 \cos 5\alpha \cos 2\alpha \sin \alpha = \sin 8\alpha - \sin 6\alpha + \sin 4\alpha - \sin 2\alpha$

b $4 \sin 5\theta \cos 3\theta \sin \theta = \cos \theta - \cos 3\theta + \cos 7\theta - \cos 9\theta$

Products to Sums Formulae:

14 PRODUCTS TO SUMS

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$
$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$
$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$
$$-2 \sin A \sin B = \cos (A + B) - \cos (A - B)$$

Mathematical Induction:

1 Prove by mathematical induction that for all positive integer values of n:

a $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

b $1 + 7 + 7^2 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$

c $1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n + 4) = \frac{1}{6}n(n + 1)(2n + 13)$

d $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots + \frac{1}{(n + 1)(n + 2)} = \frac{n}{2(n + 2)}$

e $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n + 2}{2^n}$

2 Prove these results by mathematical induction:

a $7^{2n-1} + 5$ is divisible by 12, for all integers $n \geq 1$,

b $2^{2n} + 6n - 1$ is divisible by 9, for all integers $n \geq 0$,

c $2^{2n+2} + 5^{2n-1}$ is divisible by 21, for all integers $n \geq 1$,

d $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9, for all integers $n \geq 0$.

3 **a** Copy and complete the table of values to the right. Then make a conjecture about the largest number that $2^{3n} - 3^n$ is divisible by, for all whole numbers $n \geq 0$.

n	0	1	2	3
$2^{3n} - 3^n$				

b Prove your conjecture by mathematical induction.

4 Prove by mathematical induction that for all positive integer values of n:

a $\sum_{r=1}^{n} r \times r! = (n + 1)! - 1$ **b** $\sum_{r=1}^{n} \frac{r - 1}{r!} = 1 - \frac{1}{n!}$

What is the limiting sum of the series in part **b**?

5 Prove that $1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$, for all integers $n \geq 1$.

(Hint: Use the factorisation $6k^3 + 15k^2 + 11k + 2 = (k + 1)(6k^2 + 9k + 2)$.)

6 Prove that $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$, for all integers $n \geq 1$.

(Hint: Use the factorisation $2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k + 1)^2(2k^2 + 4k + 1)$.)

Mathematics Advanced

Mathematics Extension 1

Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

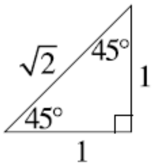
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



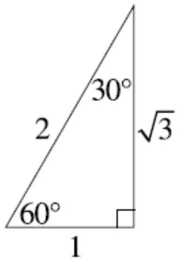
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

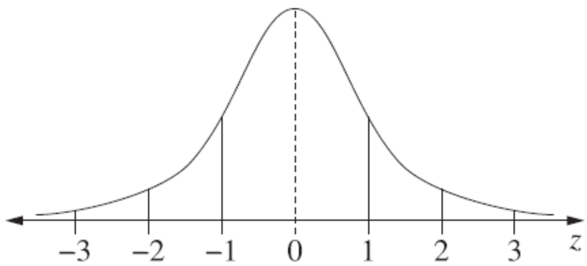
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

ANSWERS:

Further Rates

Chapter 16 review exercise

1 a $600\ \text{mm}^2/\text{h}$.

b $3\sqrt{2}\,\text{mm/h}$ (The rate is constant.)

2 a $h = r\sqrt{3}$ **b** $\frac{V}{t} = \pi r^2\sqrt{3}\frac{r}{t}$

c $\ell = 2r$ **d** $\frac{A}{t} = 4\pi r\frac{r}{t}$ **f** $\frac{5}{\sqrt{3}}\text{m}^2/\text{s}$

3 b $k = \frac{1}{5}\log_e\frac{13}{8}$, $80000e^{18k} \doteqdot 459000$

c $t = \frac{1}{k}\log_e 12.5$, year 2036

4 b $k = \frac{\log_e 2}{30.2}$ **c** 10.07% **d** about 201 years

5 a The temperature is dropping, but $T - E$ is positive.

c i $A = 500$, $k = -\frac{1}{6}\log_e 2$, $T = 500e^{15k} \doteqdot 88°C$

ii $A = 460$, $k = -\frac{1}{6}\log_e\frac{46}{21}$,

$T = 40 + 460e^{15k} \doteqdot 105°\text{C}$

6 a The population is growing, and $P - M$ is negative.

b As $t \to \infty$, $P \to M - 0 = M$.

c i $A = 9500$, $k = \frac{1}{10}\log_e\frac{19}{16}$

ii $p = 10000 - 9500e^{-20k} \doteqdot 3260$

iii $t = \frac{1}{k}\log_e\frac{19}{4} \doteqdot 91$ years, year 2100.

Further Trigonometry

Chapter 17 review exercise

1 a $-1 \le f(x) \le 4$

b $f^{-1}(x) = 2x - 2$

c $-1 \le x \le 4$, $-4 \le f^{-1}(x) \le 6$

d $y = x$

2 a $x > -1$, all real y

b $F^{-1}(x) = e^x - 1$

c All real x, $y > -1$

e Both functions are increasing.

3 b $x \le 2$

c $Q^{-1}(x) = 2 - \sqrt{x}$, because $Q^{-1}(x) \le 2$.

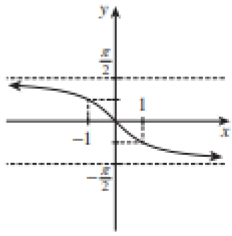
4 a $\frac{\pi}{2}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{3}$ **d** $-\frac{\pi}{4}$ **e** $\frac{2\pi}{3}$ **f** $-\frac{\pi}{6}$

5 a 1 **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{1}{2}$ **d** $-\frac{\pi}{6}$ **e** $\frac{2\pi}{3}$ **f** $-\frac{\pi}{3}$

6 a $\frac{2\sqrt{2}}{3}$ **b** $\frac{3}{4}$

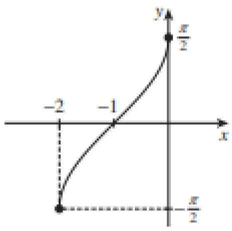
7 a All real x,

$-\frac{\pi}{2} < y < \frac{\pi}{2}$



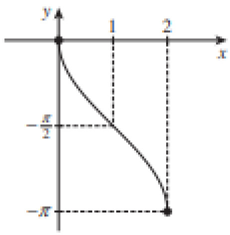
b $-2 \le x \le 0$,

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



c $0 \le x \le 2$,

$-\pi \le y \le 0$



8 a $\cos 2\theta$ **b** $\sin 40°$ **c** $\tan 50°$

d $\cos 70°$ **e** $\sin 6\alpha$ **f** $\frac{1}{\tan\theta} = \cot\theta$

9 a $\sin 4\theta$ **b** $\cos x$ **c** $\cos 6\alpha$

d $\tan 70°$ **e** $\cos 50°$ **f** $\tan 8x$

10 a $\frac{4}{5}$ **b** $\frac{7}{25}$ **c** $-\frac{16}{65}$ **d** $\frac{120}{169}$ **e** $\frac{24}{7}$ **f** $\frac{33}{56}$

12 a $\sin 30° = \frac{1}{2}$ **b** $\cos 30° = \frac{\sqrt{3}}{2}$

c $\tan 135° = -1$ **d** $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

e $\frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$ **f** $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

15 b $\frac{\pi}{4}$

18 a $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$

b $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$

c $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

d $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ or 2π

19 a $\frac{2t}{1 + t^2}$ **b** $\frac{1 - t^2}{1 + t^2}$ **c** $\frac{2t}{1 - t^2}$

d $\frac{1 + t^2}{1 - t^2}$ **e** $\frac{2}{1 + t^2}$ **f** t

20 a $\tan 135° = -1$

b $\sin 135° = \frac{1}{\sqrt{2}}$

c $\cos 135° = -\frac{1}{\sqrt{2}}$

22 b ii $-2 + \sqrt{3} = \tan 165°$, because

$\tan 330° = -\tan 30° = -\frac{1}{\sqrt{3}}$.

23 a $\cos 3A + \cos A$

b $\sin 8A - \sin 2A$

c $\sin(5A + B) + \sin(A + 3B)$

d $\cos(A + 2B) - \cos(3A + 8B)$

Mathematical Induction

Chapter 2 review exercise

3 a 0, 5, 55, 485. . . .

The expression is always divisible by 5.

4 b The limiting sum is 1