Name: ____________________

Year 12 2024/2025 Extension 2 Mathematics Assessment Task 1

Investigative Assignment with Validation Task

Task number: 1	**Weighting:** 20%	**Due Date:** Wednesday 11/12/24

Outcomes assessed:

MEX12-1	understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
MEX12-2	chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-4	uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-7	applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
MEX12-8	communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Nature and description of the task:

As a result of completing this Investigative Assignment, students should be familiar with complex numbers including all operations, square roots, polar form and its properties, Euler's Formula and applications of De Moivre's Theorem. They will also be able to solve quadratic and polynomial equations with complex co-efficients, find the roots of unity, perform operations on the complex plane as well as sketching curves and regions.

The Assignment will consist of two parts:

- Part 1 Preparation Activity (value >> 50% of the overall Assignment) – completed at home. The suggested time for the Preparation Activity is one week, although you will have 2 weeks to complete it. All answers will need to be completed on your own paper and attached to this booklet. The marks allocated to each question is shown next to the question. All necessary working will need to be shown, and answers/responses should be correct/detailed to obtain full marks.
- Part 2 Validation Task (value >> 50% of the overall Assignment) – On the 11th December, 2024 you will receive a selection of similar questions to the Preparation Activity to be completed in a 1 hour in-class Validation Task. You will hand in the answers to the questions of the Preparation Activity at the start of the Validation Task AND therefore you will NOT have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to hand in / complete both parts of the Assignment on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is submitted/completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity (93 Marks) – attempt each of the following questions on your own paper in preparation for the in-class Validation Task.

Complex Numbers:

		Marks
1	If $z = 2 + 3i$ and $w = 10 - 4i$, find:	
	(a) $5z - \bar{w}$	1
	(b) z^3	2
	(c) $\frac{w}{z}$	2
2	Write as a product of two complex linear factors: $z^2 + 5z + 74$	2
3	Solve the following equation for z: $2z^2 + 3z + 5 = 0$	2
4	(a) Find the square roots of $-3 + 4i$.	2
	(b) Hence, or otherwise, solve the following quadratic equation for z in the form of $a + ib$ where a and b are real numbers: $z^2 + z + 1 - i = 0$	2
5	It is known that $1 - 4i$ is a zero of the polynomial $P(z) = z^3 - 8z^2 + 29z - 102$.	
	(a) Find the zeroes of $P(z)$.	2
	(b) Hence, write $P(z)$ as a product of two factors with real coefficients.	1
6	Express the following complex number in modulus-argument form: $-\sqrt{3} - 3i$	2
7	Express the following complex number in Cartesian form: $3\text{cis}\frac{2\pi}{3}$	2
8	(a) Show that $1 - \sqrt{3}i$ in modulus-argument form is $2\text{cis}\frac{-\pi}{3}$.	2
	(b) Hence, simplify in modulus-argument form $(1 - \sqrt{3}i)^5$.	1
	(c) Express in Cartesian form $(1 - \sqrt{3}i)^5$.	1
9	Simplify the following:	
	(a) $2\text{cis}\frac{\pi}{4} \times 5\text{cis}\frac{\pi}{3}$	1
	(b) $\frac{12\text{cis}4\theta}{3\text{cis}5\theta}$	1
10	Simplify: $(1 + i\tan\theta)^3$	2
11	(a) Sketch the graph in the complex plane represented by $\text{Arg}\left(\frac{z-2}{z+2}\right) = \frac{\pi}{2}$	2
	(b) The graph of $\lvert z\rvert = \lvert z+1\rvert$ intersects with $\text{Arg}\left(\frac{z-2}{z+2}\right) = \frac{\pi}{2}$ at a point P. Find the complex number represented by the point P.	2
12	Sketch the region in the complex plane that simultaneously satisfies: $0 < \text{Arg}(z + 3) < \frac{\pi}{6}$ and $\text{Re}(z) \leq 1$.	3
13	(a) Express $\frac{1+i\sqrt{3}}{1+i}$ in modulus-argument form.	2
	(b) Hence, find the exact value of $\cos\frac{\pi}{12}$.	2
14	Sketch the graph specified by the equation: $z\bar{z} - 3(z + \bar{z}) = 7$	3

15	For the complex numbers $z = 6\text{cis}\frac{\pi}{6}$ and $w = 2\text{cis}\frac{5\pi}{3}$, plot the point represented by $\frac{iz}{w}$.	2
16	If $z = \text{cis}\frac{2\pi}{3}$ and $w = 2i$, find the value of $\arg(z+w)$.	2
17	Given $0 < \theta < \frac{\pi}{2}$, find $\arg\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)$.	3
18	Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)^4(\cos\theta + i\sin\theta)^5}{(\cos 3\theta + i\sin 3\theta)^2}$	2
19	Evaluate $\dfrac{(e^{i\frac{\pi}{4}})^3}{(e^{i\frac{\pi}{5}})^7}$	2
20	(a) Write $-2+2i$ in mod-arg form. (b) Hence find $(-2+2i)^5$ in Cartesian form.	2 2
21	By using De Moivre's Theorem to find $\cos 7\theta$ and $\sin 7\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$, find an expression for $\tan 7\theta$ in terms of powers of $\tan\theta$.	4
22	Solve $z^3 + 8 = 0$ with roots in the form $re^{i\theta}$.	2
23	Find the five fifth roots of -1 in the form $re^{i\theta}$.	2
24	(a) Express $\dfrac{1+i\tan\theta}{1-i\tan\theta}$ in $re^{i\theta}$ form. (b) Hence, or otherwise, find the three cube roots of $\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}$ in $re^{i\theta}$ form.	2 2
25	(a) Show that $e^{-ix} = \cos x - i\sin x$. (b) Hence, show that $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$. (c) Find the value of $\cos(2i)$.	1 2 1
26	(a) Show that $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$. (b) Solve for $x \in \mathbb{C}$: $\sin x = 2$.	2 3
27	Show that one of the values of i^i is $\dfrac{1}{\sqrt{e^{\pi}}}$. (There are infinitely many values for i^i)	2
28	By expanding $\left(z - \frac{1}{z}\right)^3$, show that: $$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$	3
29	(a) By using De Moivre's Theorem, show that $$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$ (b) Let $z = \cos\theta + i\sin\theta$. Show that: $$16\sin^3\theta\cos^2\theta = 2\sin\theta + \sin 3\theta - \sin 5\theta$$ (c) Express $\sin 5\theta + 9\sin 3\theta$ as a polynomial in $\sin\theta$. (d) Hence solve $\sin 5\theta + 9\sin 3\theta - 8\sin\theta = 0$ for $0 \le \theta \le \pi$.	3 3 2 2

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

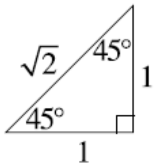
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

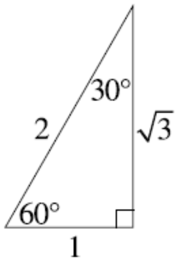


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\text{cosec}\, A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

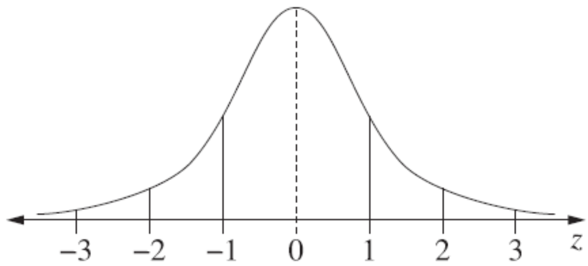
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\text{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \text{Bin}(n, p)$$

$$\Rightarrow \ P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^ne^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$