Name: ____________________

Year 12 2019 Extension 2 Mathematics Assessment Task 3

Investigative Assignment with Validation Task		
Task number: 3	**Weighting:** 25%	**Due Date:** Wednesday 26/6/19

Outcomes assessed:

- E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- E6 combines the ideas of algebra and calculus to determine the important measures of the graphs of a wide variety of functions
- E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae to problems
- E9 communicates abstract ideas and relationships using appropriate notation and logical argument

Nature and description of the task:

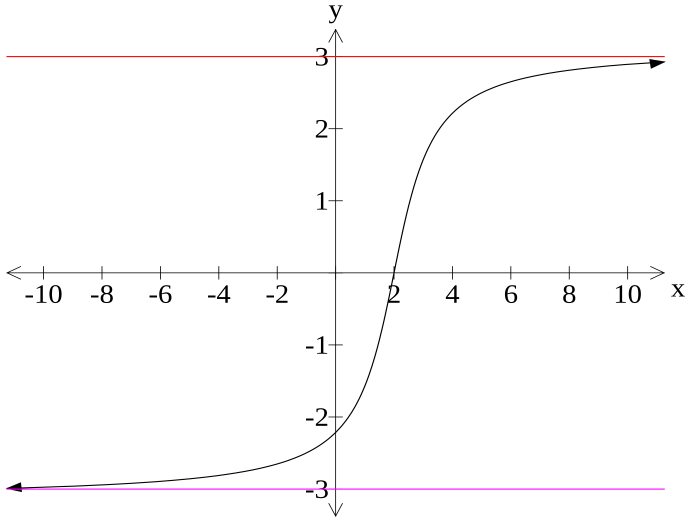
On the 26th June, 2019 you will receive a selection of the following questions from the Preparation Activity below to complete in 50 minutes in an in-class Validation Task. You are expected to investigate/attempt each of these questions before the in-class Validation Task. The final mark for this assessment will be the mark you receive on the in-class Validation task. NOTE: You will not have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to hand in / complete both parts of the Assignment on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is submitted/completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity – investigate/attempt each of the following questions in preparation for the in-class Validation Task.

1) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x+2)$ **1**

(ii) $y = \left| f(x) \right|$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = f(|x|)$ **2**

2) If α is the complex number $-2 + 2\sqrt{3}i$, find

i) $|\alpha|$ 1

ii) $\arg \alpha$ 1

iii) Show that $\alpha^2 = 4\overline{\alpha}$ 2

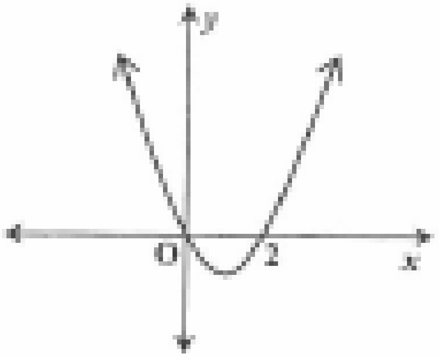
iv) Show that α is a root of the equation $w^3 - 64 = 0$

and find the other roots 3

3) i) Express $z = \dfrac{1 + \sqrt{3}i}{1 + i}$ in the form $r(\cos\theta + i\sin\theta)$ 3

ii) Find the smallest positive integer n such that z^n is a real number 1

4) Given $f(x) = x^2 - 2x$. On separate diagrams sketch the graphs of the following. Indicate clearly any asymptotes, intercepts with the axes and local maxima and minima.



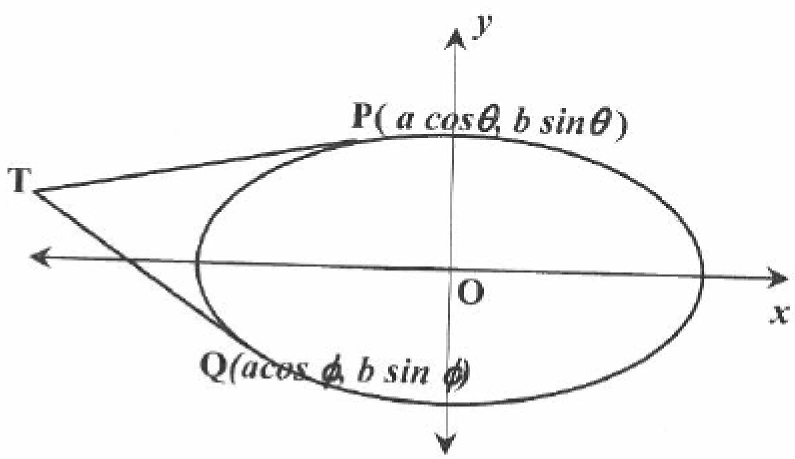
i)	$y = \left\| f(x) \right\|$	**1**
ii)	$y = f\left(\|x\|\right)$	**1**
iii)	$y = \dfrac{1}{f(x)}$	**2**
iv)	$y = e^{f(x)}$	**1**
v)	$y^2 = f(x)$	**1**
vi)	$y = \left[f(x) \right]^2$	**1**
vii)	$y = \ln\left[f(x) \right]$	**2**

5) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2 + 1}}\, dx$. **2**

6) i) Show that $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ **2**

ii) Show that $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$ **2**

iii) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. TQ and TP are tangents to the ellipse as shown in the diagram below.

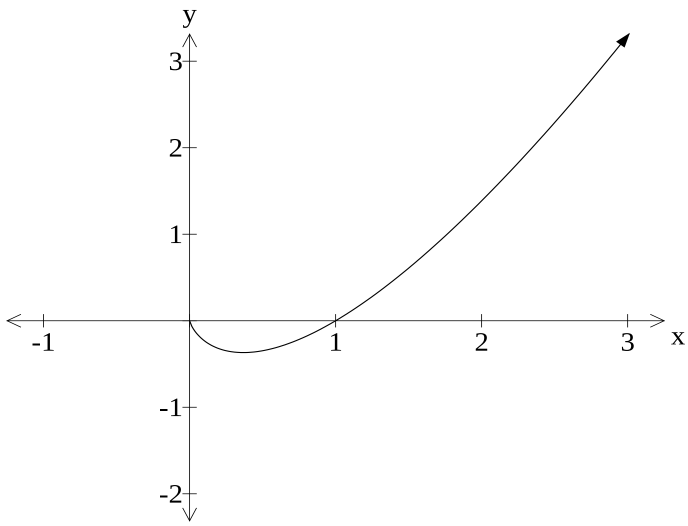


α) Show that T is the point $\left[\dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}, \dfrac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right]$ **3**

β) Find the locus of T if $\theta - \phi = \frac{\pi}{2}$ **2**

7) Evaluate $\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}}\, dx$. **2**

8) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \left| f(x) \right|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = (f(x))^2$ **2**

9) i) Solve $z^5 = -1$ **2**

ii) Hence, show that $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$ **3**

10) Use integration by parts to find $\int \tan^{-1} x\ dx$. **3**

11) A hyperbola has eccentricity $\frac{5}{4}$ and foci (-5, 0) and (5, 0). **3**

Find the equation of the hyperbola.

12) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent.

i) Show that $ST = \dfrac{|e\cos\theta - 1|}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$ **2**

ii) Hence, prove that $ST \times S'T' = b^2$ **3**

13) If $\arg z_1 \neq \arg z_2$, show that $|z_1| + |z_2| > |z_1 + z_2|$ **2**

14) A point $P\ (a\sec\theta,\ b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with foci $S\ (ae,\ 0)$ and $S'\ (-ae,\ 0)$.

i) Show that $PS = a(e\sec\theta - 1)$ and $PS' = a(e\sec\theta + 1)$ **3**

ii) By considering when P is on both branches of the hyperbola, show that $|PS - PS'| = 2a$ **3**

15) The complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1+2i}$ where a and b are real, are such that $z_1 + z_2 = 1$. Find a and b. **3**

16) (i) Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. Find z^6. **2**

(ii) Plot on the Argand diagram, all complex numbers that are solutions of $z^6 = -1$. **2**

17) (i) Express each of $z_1 = \sqrt{3}+i$ and $z_2 = -\sqrt{2}+\sqrt{2}i$ in modulus-argument form. **2**

(ii) Find the exact value of $\arg\left(\frac{z_2}{z_1}\right) - \arg(z_1 + z_2)$. **2**

18) The point P representing the complex number z moves on the Argand diagram so that $|z| = |z - 6 + 4i|$.

(i) Show that the locus of P has the equation $3x - 2y - 13 = 0$. **2**

(ii) Hence find the minimum value of $|z|$. **1**

19) (i) Find $\int \frac{dx}{x^2 + 2x + 5}$. **2**

(ii) Hence, or otherwise, find $\int \frac{x^2}{x^2 + 2x + 5}\, dx$. **3**

20) (i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $P\left(2t, \frac{2}{t}\right)$ has the equation $x + t^2 y - 4t = 0$. **2**

(ii) The tangent at P cuts the x axis at point Q. Show that the line through Q which is perpendicular to the tangent at P has the equation $t^2 x - y - 4t^3 = 0$. **1**

(iii) The line $t^2 x - y - 4t^3 = 0$ cuts the rectangular hyperbola at the points R and S. Show the midpoint M of RS has coordinates $M(2t, -2t^3)$. **2**

(iv) Find the equation of the locus of M as P moves on the rectangular hyperbola. State any restrictions. **2**

21) Point $P(x_0, y_0)$ is on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

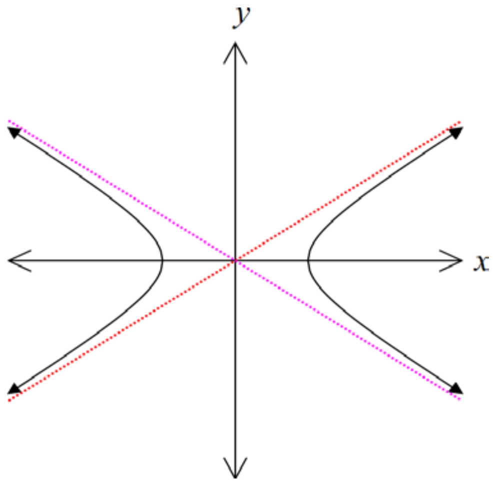
(i) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(ii) Show the equation of the tangent at P is $\frac{x_0 x}{25} + \frac{y_0 y}{9} = 1$. **2**

(iii) Let the tangent at P meet a directrix at a point Q. **2**

Show that $\angle PSQ$ is a right angle, where S is the corresponding focus.

22) The point $P(x_0, y_0)$ lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$.



(i) What are the equations of the asymptotes of the hyperbola? **1**

(ii) Show that the acute angle θ between the two asymptotes **2**

satisfies $\tan\theta = \dfrac{2ab}{a^2 - b^2}$

(iii) If C and D are the feet of the perpendicular drawn for **3**

$P(x_0, y_0)$ to the asymptotes show that $CP \times DP = \dfrac{a^2b^2}{a^2 + b^2}$

(iv) What is the area of ΔPCD? **2**

23) Find the indefinite integral $\displaystyle\int \frac{(3\tan x + 1)\sec^2 x}{(\tan x + 1)^2}\,dx$ by letting $u = \tan x + 1$. **3**

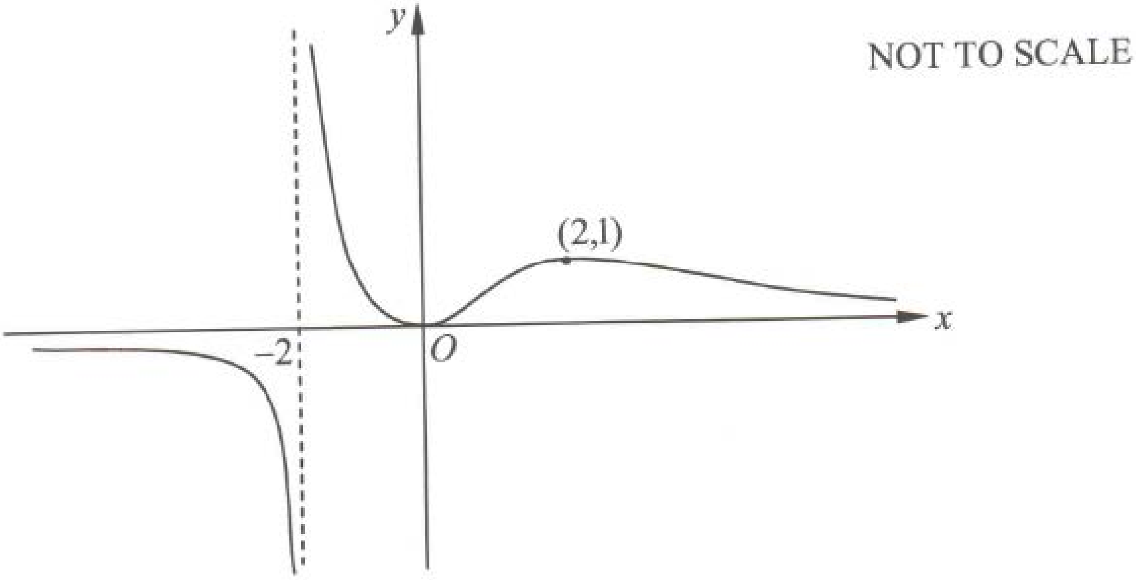
24) By using t- results, find the primitive of $\displaystyle\int \frac{dx}{4 + 3\cos x}$. **3**

25) Tangents to the ellipse $16x^2 + 25y^2 = 400$ at $P(5\cos\alpha, 4\sin\alpha)$ and $Q(5\cos\beta, 4\sin\beta)$ are at right angles to each other.

(i) Show that the gradient of the tangent at P is $-\dfrac{4\cos\alpha}{5\sin\alpha}$ **2**

(ii) Hence show that $25\tan\alpha\tan\beta = -16$. **2**

26) The function defined by $y = f(x)$ is given below



Draw separate, one-third page sketches of the following

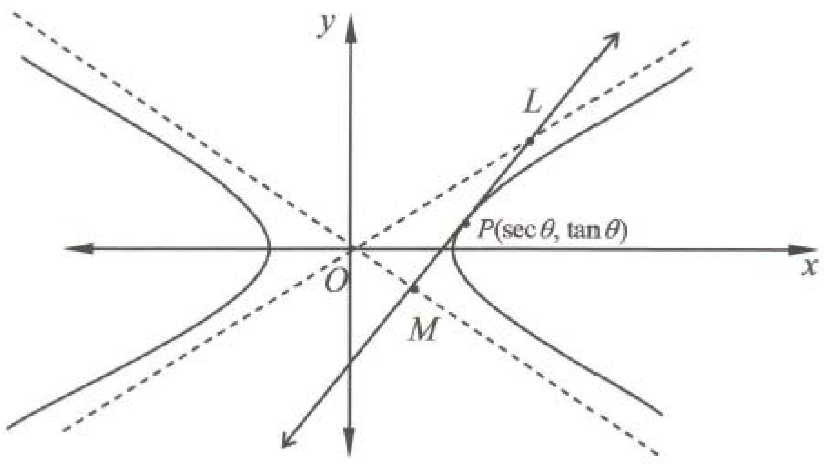
i) $y = f(-x)$ **2**

ii) $y = \dfrac{1}{f(x)}$ **2**

iii) $y = \left[f(x) \right]^2$ **2**

27) The equation of a curve is $x^2 + 3xy + 4y^2 = 58$. Find the equation of the normal to the curve at the point (2, 3) **3**

28) The diagram shows the hyperbola with the equation $x^2 - y^2 = 1$



i) Show that the equation of the tangent to this hyperbola at $P\left(\sec\theta,\ \tan\theta\right)$ **2**

is given by $x\sec\theta - y\tan\theta = 1$

ii) This tangent cuts the asymptotes at L and M. Show that *LP* = *PM*. **3**

iii) Show that the area of ΔOLM , where O is the origin, is independent **3**

‘ of the position of P.

29) Evaluate $\int_0^{\frac{\pi}{3}} \sec^4 x \ dx$. **3**

30) Use partial fractions to find $\int \frac{2y+3}{(y-2)(y^2+3)}\ dy$. 4

END OF POSSIBLE IN-CLASS VALIDATION TASK QUESTIONS