Name: ___________________

Year 12 2020 Mathematics Extension 1
Assessment Task 3

<table>
<tr><td colspan="3">Assignment – Differential Equations</td></tr>
<tr><td>Task number: 3</td><td>Weighting: 25%</td><td>Due Date: Wednesday 5/8/20</td></tr>
<tr><td colspan="3">Outcomes assessed:

ME11-4 applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change
ME12-1 applies techniques involving proof or calculus to model and solve problems
ME12-4 uses calculus in the solution of applied problems, including differential equations and volumes of solids of revolution
ME12-6 chooses and uses appropriate technology to solve problems in a range of contexts
ME12-7 evaluates and justifies conclusions, communicating a position clearly in appropriate mathematical forms</td></tr>
<tr><td colspan="3">Nature and description of the task:

This Assignment provides opportunities for students to develop skills, including the use of technology, to graph slope fields and solve differential equations. Practical applications of differential equations will also be investigated.

The Assignment will consist of two parts:
• Part 1 Preparation Activity (value >> 50% of the overall Assignment) – completed at home. The suggested time for the Preparation Activity is one week, although you will have approximately 4 weeks to complete it. All answers will need to be completed on your own paper and attached to this booklet. The marks allocated to each question is shown next to the question. All necessary working will need to be shown and answers /responses should be correct / detailed to obtain full marks. Use technology to complete the learning tasks in Section A and complete Section B without technology, showing full working out.
• Part 2 Validation Task (value >> 50% of the overall Assignment) – to be conducted in class for a period of 50 minutes. The Preparation Activity cannot be used during the Validation Task and will be handed in together with the Validation Task at the conclusion of the task. Technology, other than calculators, will not be used, and all marks for each question will be clearly shown next to each question on the task.</td></tr>
<tr><td colspan="3">Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to hand in / complete both parts of the Assignment on the due day, then you must have supportive documentation. Zero marks will apply if the Assessment Task is submitted/completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.</td></tr>
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Part 1 Preparation Activity (60 Marks)

Section A: (20 Marks)

Software you'll need

1. **GeoGebra:** Visit www.geogebra.org and under "Ready for Tests", click on "GeoGebra Classic." GeoGebra should open up in your browser. You can also download a desktop version by clicking on https://www.geogebra.org/download?lang=en-AU and then scrolling down to "GeoGebra Classic 6" and clicking "DOWNLOAD." Either one you use will be fine, though the instructions in this document will look more like the online version. However, the desktop version is more reliable and powerful.
2. **Document Editor:** For this assignment, you will also need a document editor, such as Microsoft Word, Google Docs, etc. Just pick your favourite one.

How to insert GeoGebra images into your report

- Use screen capture software or plugins (for example, on Windows 10 you can use the built-in snipping tool by holding down Windows key, Shift, and S), and then paste the image into your favourite document editor.
- GeoGebra Classic online version: Download the image by going to the hamburger menu() on the top right, then "File," then "Download as…," then "png." Then drag the PNG file into your favourite document editor.
- GeoGebra Classic 6 desktop version: Go to "Edit," then "Graphics View to Clipboard," then paste the image into your favourite document editor (usually Ctrl-V, or Edit->Paste).

Important things to know for this assignment

- Here's how you can draw a slope field for $y' = \cos\left(\frac{x}{4}\right) + \frac{y}{4}$:

  `f(x,y)=cos(x/4)+y/4` ← This defines $f(x,y)$ to be $\cos\left(\frac{x}{4}\right) + \frac{y}{4}$

  `SlopeField(f,30)` ← This shows a slope field for $\frac{dy}{dx} = f(x,y)$. 30 represents the number of slope segments to draw vertically and horizontally.

  `SolveODE(f, (1,0))` ← This shows the particular solution for $\frac{dy}{dx} = f(x,y)$ and $y(1) = 0$

- Here are a couple of ways you can adjust the x scale and y scale:
  - Click on and then . Under the "Basic" tab, you can adjust the minimum and maximum x and y values that you see.
  - Hold shift and drag x-axis or y-axis to adjust the scale.
  - Resize your browser or desktop application window.

- If you need help, you can find GeoGebra tutorials here: https://wiki.geogebra.org/en/Tutorials

Question 1 Slope Fields

Consider the DE: $y' = (\sin x)(\sin y)$.

a. **Using GeoGebra**, plot a slope field and adjust the x and y scales so the window shows approximately $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$. **3**

b. **By hand**, find the particular solution that satisfies the initial value problem: $y' = (\sin x)(\sin y)$, $y(0) = \frac{\pi}{4}$. Be sure you solve explicitly for y. (Hint: $\csc x + \cot x = \frac{1}{\tan\left(\frac{x}{2}\right)}$) **3**

c. Over the slope field you plotted in part (a), **use GeoGebra** to plot the solution curve that you found in part (b) with $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$. Does the solution flow along with the slope field? **4**

Question 2 Autonomous DEs

An **autonomous** differential equation is a DE that can be written in the form $y' = f(y)$. That is, y' only depends on y, not x. So, when you plot the slope field (part (a) below), notice that if you look across the slope field horizontally (constant y), the slopes will be the same.

Use the autonomous DE $\frac{dy}{dx} = 0.01y(1-y)\left(y-\frac{1}{2}\right)$ to answer the questions below.

a. **Using GeoGebra**, plot a slope field with $0 \leq x \leq 300$ and $-2 \leq y \leq 2$, and plot solution curves with each of the following initial conditions (using the `SolveODE` command): $y(0) = -1$, $y(0) = \frac{1}{4}$, $y(0) = \frac{3}{4}$, $y(0) = 2$ **3**

b. Suppose the DE models the population of people on an island. So, y represents the population (in millions of people), x represents time (in years). Based on the model, if the population is 2 million people when $x = 0$, what will the population be 40 years later? **Use GeoGebra** to answer this question. The output of `SolveODE` will be a function, so you can just plug 40 into that function. **3**

c. Based on the model as explained in part (b), what would happen if the population started below 500,000 people? How can you tell by looking at the DE? **4**

Section B: (40 Marks)

Answer these questions without the use of Geogebra or equivalent.
Show all of your working.
Answer on your own paper.

QUESTIONS

3 Determine the solution to $\frac{dy}{dx} = 1 - e^{-x}$ that passes through the point (0, 6). **2**

4 Draw the direction field for the differential equation $\frac{dy}{dx} = xy$ and show the solution that passes through *A*(0, 1). **3**

5 Solve the differential equation $\frac{dy}{dx} = \sin^3 x \cos^3 x$ given that *y*(0) = 0. **3**

6 Solve the differential equation $\frac{dy}{dx} = y^2 + 1$, given that *y*(0) = 1. **3**

7 A glass of water at a temperature of 20°C is placed in a refrigerator. The rate at which the water's temperature decreases is proportional to the difference between its temperature and the 4°C temperature inside the refrigerator. When the glass has been in the refrigerator for 10 minutes, the temperature of the water in the glass is 12°C. What is the temperature of the water in the glass after
20 minutes? **5**

8 The rate of decay of a radioactive substance is proportional to the amount, *M*, of the substance present at any time, given by the differential equation $\frac{dM}{dt} = -kM$, where *k* is a constant.

Initially, *M* = 20 units, and when *t* = 20 hours, *M* = 5 units.

How long does it take for *M* to decay to 2 units? **3**

9 After use of an insecticide, the insect population is given by $N = 200 - \frac{150}{1+e^{-t}}$, where N is the number of insects and t is the time in hours after the insecticide has been applied.

5

a How many insects were there initially?

b Write an expression for the rate at which the population of insects is changing.

c Using this model, will the population disappear altogether?

d How long will it take for the population of insects to diminish to 100?

10 The rate at which a virus spreads throughout a population of 1000 animals in a specific area is
proportional to the product of the number N of infected animals and the number of animals not infected
after t days. Initially, just 2 animals are infected, and after 10 days, 12 are infected.

5

a Write a differential equation to show the rate at which the virus is spreading.

b Determine how many animals were NOT infected after 30 days.

11 In Newton's Law of Cooling, the rate at which an object changes temperature is proportional to the
difference between its temperature and the temperature of the surrounding medium. A lamb roast is
cooked in an oven at 200°C. Then it is placed on a cooling rack in a room where the temperature is 30°C.
After 5 minutes, the roast has cooled to 120°C.

4

a Find the temperature after 15 minutes.

b Determine when the roast reaches 60°C.

12 Suppose that *t* hours after the start of an epidemic in a small community, the number of residents who have caught the disease is given by the equation,

$$P(t) = \frac{1200}{6 + 294e^{-0.8t}}$$

7

a How many residents had the disease when the epidemic began?

b Approximately how many residents will contract the disease?

c What is the differential equation for $P(t)$?

d When was the disease the most contagious?

e How fast was the disease spreading at the peak of the epidemic?

End of Preparation Activity