Name: ___________________

Year 12 2020/2021 Extension 2 Mathematics Assessment Task 1

Investigative Assignment with Validation Task

Task number: 1	**Weighting:** 20%	**Due Date:** Monday 14/12/20

Outcomes assessed:

MEX12-1	understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
MEX12-2	chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-4	uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems
MEX12-7	applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems
MEX12-8	communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Nature and description of the task:

As a result of completing this Investigative Assignment, students should be familiar with complex numbers including all operations, square roots, polar form and its properties, Euler's Formula and applications of De Moivre's Theorem. They will also be able to solve quadratic and polynomial equations with complex co-efficients, find the roots of unity, perform operations on the complex plane as well as sketching curves and regions.

On the 14th December, 2020 during period 1 you will receive a selection of similar questions from the Preparation Activity below to complete in 50 minutes in an in-class Validation Task. You are expected to investigate/attempt each of these questions before the in-class Validation Task. The final mark for this assessment will be the mark you receive on the in-class Validation task. NOTE: You will not have access to the Preparation Activity during the Validation Task.

Non-Completion of Task:

If you know you are going to be away on the day of the Validation Task and are unable to hand in / complete both parts of the Assignment on the due day, then you must have supportive documentation. *Zero marks will apply if the Assessment Task is submitted/completed late, unless an Illness/ Misadventure or Application for Extension form has been submitted.*

Part 1 Preparation Activity – investigate/attempt each of the following questions in preparation for the in-class Validation Task.

Complex Numbers I:

1. If $z = 3 - i$ and $w = 17 + i$, find:

(a) $6z - \overline{w}$ (b) z^3 (c) $\frac{w}{z}$

2. Write as a product of two complex linear factors.

(a) $z^2 + 100$ (b) $z^2 + 10z + 34$

3. Solve each quadratic equation for z.

(a) $z^2 - 8z + 25 = 0$ (b) $16z^2 + 16z + 13 = 0$

4. Find the square roots of:

(a) $5 - 12i$ (b) $7 + 6\sqrt{2}\,i$

5. Solve for z:

(a) $z^2 - 5z + (7 + i) = 0$ (b) $z^2 - (6 + i)z + (14 + 8i) = 0$

6. If $3i$ is a zero of a polynomial $P(z)$ with real coefficients, explain why $z^2 + 9$ is a factor of $P(z)$.

7. It is known that $2 + 5i$ is a zero of the polynomial $P(z) = z^3 - 8z^2 + 45z - 116$.

(a) Why is $2 - 5i$ also a zero of $P(z)$?

(b) Use the sum of the zeroes to find the third zero of $P(z)$.

(c) Hence write $P(z)$ as a product of two factors with real coefficients.

8. Express each complex number in modulus-argument form.

(a) $1 - i$ (b) $-3\sqrt{3} + 3i$

9. Express each complex number in Cartesian form.

(a) $4\operatorname{cis}\frac{\pi}{2}$ (b) $\sqrt{6}\operatorname{cis}(-\frac{3\pi}{4})$

10. Simplify:

(a) $2\operatorname{cis}\frac{\pi}{2} \times 3\operatorname{cis}\frac{\pi}{3}$ (b) $\dfrac{10\operatorname{cis}10\theta}{5\operatorname{cis}5\theta}$ (c) $(3\operatorname{cis}3\alpha)^2$

11. Sketch the graph in the complex plane represented by the equation:

(a) $|z - 2i| = 2$ (c) $\arg(z + 2) = -\frac{\pi}{4}$

(b) $|z| = |z - 2 - 2i|$ (d) $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$

12. Shade the region in the complex plane that simultaneously satisfies $|z| \geq 1$, $\text{Re}(z) \leq 2$ and $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$.

13. Suppose that $z = -1 + \sqrt{3}\,i$ and $w = 1 + i$.

(a) Find $\dfrac{z}{w}$ in the form $a + ib$, where a and b are real.

(b) Write z and w in modulus-argument form.

(c) Hence write $\dfrac{z}{w}$ in modulus-argument form.

(d) Deduce that $\cos\frac{5\pi}{12} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$.

14. Sketch the graph specified by the equation:

(a) $z\overline{z} = z + \overline{z}$ (b) $\overline{z} = iz$ (c) $|z+2| = 2|z-4|$

15. A triangle PQR in the complex plane is isosceles, with $\angle P = 90^\circ$. The points P and Q represent the complex numbers $4 - 2i$ and $7 + 3i$ respectively. It is also known that the points P, Q and R are in anticlockwise order. Find the complex numbers represented by:

(a) the vector PQ, (b) the vector PR, (c) the point R.

16. If $z_1 = 4 - i$ and $z_2 = 2i$, find in each case the two possible values of z_3 so that the points representing z_1, z_2 and z_3 form an isosceles right-angled triangle with the right-angle at:

(a) z_1 (b) z_2

17. In an Argand diagram, O is the origin and the points P and Q represent the complex numbers z_1 and z_2 respectively.

If triangle OPQ is equilateral, prove that ${z_1}^2 + {z_2}^2 = z_1 z_2$.

18. If $z_1 = 2\operatorname{cis}\frac{\pi}{12}$ and $z_2 = 2i$, find:

(a) $\arg(z_1 + z_2)$ (b) $\arg(z_2 - z_1)$

19. If z_1 and z_2 are complex numbers such that $|z_1| = |z_2|$, prove that

$$\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right).$$

20. If $z = \operatorname{cis}\theta$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$.

21. The points A, B, C and O represent the numbers z, $\dfrac{1}{z}$, 1 and 0 respectively in the complex plane. Given that $0 < \arg z < \frac{\pi}{2}$, prove that $\angle OAC = \angle OCB$.

22. (a) By drawing a suitable diagram, prove the triangle inequality $|z_1 - z_2| \geq |z_1| - |z_2|$.

(b) Hence find the maximum value of $|z|$ given that $\left|z - \dfrac{4}{z}\right| = 2$.

Complex Numbers II – De Moivre and Euler:

1. Simplify:

(a) $(\cos\theta + i\sin\theta)^3(\cos 2\theta + i\sin 2\theta)^2$ (b) $\dfrac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^2}$

2. Evaluate $\dfrac{\left(e^{-i\frac{\pi}{7}}\right)^3}{\left(e^{i\frac{\pi}{7}}\right)^4}$.

3. (a) Write $1 - i$ in mod-arg form.

(b) Hence find $(1-i)^{13}$ in Cartesian form.

4. (a) Use de Moivre's theorem to evaluate $(\sqrt{3}+i)^{12} + (\sqrt{3}-i)^{12}$.

(b) If n is a positive integer:

(i) prove that $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ is real,

(ii) determine the values of n for which $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ is rational.

5. (a) Use de Moivre's theorem to find $\cos 6\theta$ and $\sin 6\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(b) Hence show that $\tan 6\theta = \dfrac{2t\left(3 - 10t^2 + 3t^4\right)}{1 - 15t^2 + 15t^4 - t^6}$, where $t = \tan\theta$.

6. (a) Expand $\left(z + \dfrac{1}{z}\right)^4$ and $\left(z - \dfrac{1}{z}\right)^4$.

(b) By letting $z = \cos\theta + i\sin\theta$, prove that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$.

7. Suppose that ω is a complex cube root of -1.

(a) Show that the other complex root is $-\omega^2$.

(b) Evaluate $(6\omega + 1)(6\omega^2 - 1)$.

8. Solve the equation $z^3 - 8i = 0$, writing the roots in the form $re^{i\theta}$.

9. Find, in mod-arg form:

(a) the three cube roots of $2 + 2i$, (b) the six sixth roots of i.

10. Suppose that $z = 4\sqrt{3}e^{i\pi/3} - 4e^{5i\pi/6}$.

(a) Simplify z, writing your answer in exponential form.

(b) Show that $\dfrac{z}{8} + i\left(\dfrac{z}{8}\right)^2 + \left(\dfrac{z}{8}\right)^3 = 2i$.

(c) Find the three cube roots of z in exponential form.

11. (a) Show that $(z - z^{-1})^7 = (z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})$.

(b) If $z = \cos\theta + i\sin\theta$, show that $z - z^{-1} = 2i\sin\theta$ and that $(z^n - z^{-n}) = 2i\sin n\theta$.

(c) Hence prove that $\sin^7\theta = \frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$.

(d) Find $\displaystyle\int \left(35\sin\theta - 64\sin^7\theta\right)\, d\theta$.

12. (a) Use de Moivre's theorem to prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Hence solve the equation $16x^4 - 20x^2 + 5 = 0$, giving the roots in trigonometric form.

(c) Show that $\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$.

(d) If $u = 2x^2 - 1$, show that $4u^2 - 2u - 1 = 0$.

(e) Deduce the exact value of $\cos\frac{\pi}{5}$.

13. (a) Derive the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7.

(b) Use these results to verify each trigonometric identity.

(i) $2\cos^2\theta = 1 + \cos 2\theta$ (iii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(ii) $2\sin^2\theta = 1 - \cos 2\theta$ (iv) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

14. (a) Find the seven seventh roots of -1 in mod-arg form.

(b) Hence show that:

(i) $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

(ii) $z^7 + 1 = (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$

(iii) $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$
$= \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$

(c) Divide both sides of the identity in (b)(iii) by z^3, and hence show that:

$$2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1 = 8\left(\cos\theta - \cos\tfrac{\pi}{7}\right)\left(\cos\theta - \cos\tfrac{3\pi}{7}\right)\left(\cos\theta - \cos\tfrac{5\pi}{7}\right)$$

15. (a) Find the fifth roots of unity in exponential form.

(b) Let α be the complex fifth root of unity with the smallest positive argument, and suppose that $u = \alpha + \alpha^4$ and $v = \alpha^2 + \alpha^3$.

(i) Find the values of $u + v$ and $u - v$.

(ii) Deduce that $\cos\frac{2\pi}{5} = \frac{1}{4}\left(\sqrt{5} - 1\right)$.

16. Let $z = \cos\theta + i\sin\theta$ and suppose that n is a positive integer.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Prove that $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

(c) Hence show that $\left(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}\right)\sin\theta = \sin(2n+1)\theta$.

(d) Use the previous part and the result $\cos 3A = 4\cos^3 A - 3\cos A$ to prove the identity:

$$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$$

17. Use the exponential forms of $\cos\theta$ and $\sin\theta$ given in Box 7 to verify that

$$\sin\alpha - \sin\beta = 2\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

18. Suppose that n is an integer greater than 2 and ω is an nth root of unity, where $\omega \neq 1$.

(a) By expanding the left-hand side, show that

$$\left(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1}\right)(\omega - 1) = n.$$

(b) Using the identity $\dfrac{1}{z^2 - 1} = \dfrac{z^{-1}}{z - z^{-1}}$, or otherwise, prove that

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}.$$

(c) Hence, if $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, find the real part of $\dfrac{1}{\omega - 1}$.

(d) Deduce that $1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5} = -\frac{5}{2}$.

(e) By expressing the left-hand side of the result in part (iv) in terms of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$, find the exact value of $\cos\frac{\pi}{5}$.

Box 7 - Exponential Forms of Sin and Cos:

THE EXPONENTIAL FORMS OF SIN AND COS: Euler's formula can be used to write these functions in exponential form. They are:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$